Vector-valued modular forms with an unnatural boundary

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Dedicated to the memory of Leon Ehrenpreis.

Abstract

We characterize all logarithmic, holomorphic vector-valued modular forms which can be analytically continued to a region strictly larger than the upper half-plane. (MSC2010: 11F12, 11F99.)

1 Introduction

Set $\Gamma = SL_2(\mathbb{Z})$ be the modular group with standard generators

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let $\rho : \Gamma \to GL(p, \mathbb{C})$ be a p-dimensional representation of Γ. A holomorphic vector-valued modular form of weight $k \in \mathbb{Z}$ associated to ρ is a holomorphic function $F : \mathfrak{H} \to \mathbb{C}^p$ defined on the upper half-plane $\mathfrak{H}$ which satisfies

$$F|_k \gamma(\tau) = \rho(\gamma) F(\tau) \quad (\gamma \in \Gamma) \tag{1}$$

and a growth condition at ∞ (see below). As usual, the stroke operator here is defined as

$$F|_k \gamma(\tau) = (c\tau + d)^{-k} F(\gamma \tau) \ \ (\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma).$$

We also refer to the pair (ρ, F) as a vector-valued modular form, and call p the *dimension* of (ρ, F). We usually consider F as a vector-valued function[1] $F(\tau) = (f_1(\tau), \ldots, f_p(\tau))^t$ and call the $f_i(\tau)$ the *component functions* of F. The purpose of the present paper is to investigate whether $F(\tau)$ has a *natural boundary*.

If $f(\tau)$ is a nonconstant (scalar) modular form of weight k on a subgroup of finite index in Γ, then it is well known that the real axis is a *natural boundary* for $f(\tau)$. That is, there is *no* real number r such that $f(\tau)$ can be analytically continued to a region containing $\mathfrak{H} \cup \{r\}$. In [KM2] the authors extended this result to *normal* vector-valued modular forms on Γ, showing that each nonconstant component of such a modular form has the real line as natural boundary. Here we study the same question for the larger class of *polynomial* (or *logarithmic*) vector-valued modular forms introduced in [KM3], where one assumes only that the eigenvalues of $\rho(T)$ have absolute value 1. This case is more subtle because the existence of a natural boundary is *no longer true* in general. Here, we say that a vector-valued modular form has the real line as a natural boundary provided that at least one component does.

Consider the column vector of polynomials

$$C(\tau) = (\tau^{p-1}, \tau^{p-2}, \ldots, 1)^t.$$

It is not hard to see (cf. Section 2) that $C(\tau)$ is a vector-valued modular form of weight $1 - p$ associated with a representation σ equivalent to the $p - 1$st symmetric power $S^{p-1}(\nu)$ of the natural defining representation ν of Γ. Thus for any p, (σ, C) is a p-dimensional holomorphic vector-valued modular form which is obviously analytic throughout the complex plane. The main result of the present paper is that this is essentially the only example of a polynomial vector-valued modular form whose natural boundary is not the real line.

In order to state our results precisely, we need one more definition. Suppose that $(\rho, F), (\rho', F')$ are two vector-valued modular forms of weight k and dimension p. We say that they are *equivalent* if there is an invertible $p \times p$ matrix A such that

$$(\rho, F) = (A\rho' A^{-1}, AF').$$

We give two formulation of the main result. As we shall explain, they are essentially equivalent.

Theorem 1. Suppose that the eigenvalues of $\rho(T)$ have absolute value 1, and let (ρ, F) be a nonzero holomorphic vector-valued modular form of weight k and dimension p. Then the following are equivalent:

> (a) $F(\tau)$ does *not* have the real line as a natural boundary,
> (b) The component functions of $F(\tau)$ span the space of polynomials
> of degree $l - 1$ for some $l \le p$. Moreover, $k = -l$.

Theorem 2. Suppose that the eigenvalues of $\rho(T)$ have absolute value 1, and let (ρ, F) be a holomorphic vector-valued modular form of weight k and dimension p.

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[1]Superscript t means *transpose* of vectors and matrices

Suppose further that the component functions of $F(\tau)$ are *linearly independent.* Then the following are equivalent:

(a) $F(\tau)$ does *not* have the real line as a natural boundary,

(b) (ρ, F) is equivalent to (σ, C) and $k = 1 - p$.

The reader familiar with Eichler cohomology will recognize the space of polynomials in τ (considered as Γ-module) as a crucial ingredient in that theory. This points to the fact that Eichler cohomology has close connections to the theory of vector-valued modular forms, connections that in fact go well beyond the question of natural boundaries that we treat here. The authors hope to return to this subject in the future.

The paper is organized as follows. In Section 2 we review some background material regarding vector-valued modular forms, discuss the basic example (σ, C) introduced above, and explain why Theorems 1 and 2 are equivalent. In Section 3 we give the proof of the Theorems.

2   Background

The space of homogeneous polynomials in variables X, Y is a right Γ-module such that $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ is an algebra automorphism with

$$\gamma : X \mapsto aX + bY, \ Y \mapsto cX + dY.$$

The subspace of homogeneous polynomials of degree $p-1$ is an irreducible Γ-submodule which we denote by Q_{p-1}. The representation of Γ that it furnishes is the $p-1$th symmetric power $S^{p-1}(\nu)$ of the defining representation ν.

For $\tau \in \mathfrak{H}$, let $P_{p-1}(\tau)$ be the space of polynomials in τ of degree at most $p - 1$. Since

$$\tau^j |_{1-p} \gamma = (c\tau + d)^{p-1} \left(\frac{a\tau + b}{c\tau + d} \right)^j = (a\tau + b)^j (c\tau + d)^{p-1-j},$$

it follows that $P_{p-1}(\tau)$ is a right Γ-module with respect to the stroke operator $|_{1-p}$. Indeed, $P_{p-1}(\tau)$ is isomorphic to Q_{p-1}, an isomorphism being given by

$$X^j Y^{p-1-j} \mapsto \tau^j \ \ (0 \le j \le p - 1).$$

Since $1, \tau, \ldots, \tau^{p-1}$ are linearly independent and span a right Γ-module with respect to the stroke operator $|_{1-p}$, we know (cf. [KM1], Section 2) that there is a unique representation $\sigma : \Gamma \to GL_p(\mathbb{C})$ such that

$$\sigma(\gamma) C(\tau) = C |_{1-p} \gamma(\tau) \ \ (\gamma \in \Gamma).$$

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This shows that (C, σ) is a vector-valued modular form of weight $1 - p$ and that the representation σ is equivalent to $S^{p-1}(\nu)$.

We can now explain why Theorems 1 and 2 are equivalent. Assume first that Theorem 1 holds. Let (ρ, F) be a holomorphic vector-valued modular form of weight k with linearly independent component functions and such that the real line is not a natural boundary for $F(\tau)$. By Theorem 1 the components of $F(\tau)$ span a space of polynomials of degree no greater than $p - 1$, and by linear independence they must span the space $P_{p-1}(\tau)$. Moreover, we have $k = 1 - p$. Now there is an invertible $p \times p$ matrix A such that $AF(\tau) = C(\tau)$, whence $(\rho, F(\tau))$ is equivalent to $(A\rho A^{-1}, C(\tau))$. As explained above, we necessarily have $A\rho A^{-1} = \sigma$ in this situation, so that (ρ, F) is equivalent to (σ, C). This shows that (a) $\Rightarrow$ (b) in Theorem 2, in which case Theorem 2 is true.

Now suppose that Theorem 2 holds, and let (ρ, F) be a nonzero holomorphic vector-valued modular form of dimension p and weight k such that the real line is not a natural boundary for $F(\tau)$. Let $(g_1, \ldots, g_l)$ be a *basis* for the span of the components of F. Setting $G = (g_1, \ldots, g_l)^t$, we again use ([KM1], Section 2) to find a representation $\alpha : \Gamma \to GL_l(\mathbb{C})$ such that (α, G) is a holomorphic vector-valued modular form of weight k. Because the components of G are linearly independent, Theorem 2 tells us that they span the space $P_{l-1}(\tau)$ of polynomials of degree at most l, and that $k = -l$. Thus the conclusions of Theorem 1(b) hold, and Theorem 1 is true.

To complete this Section we recall (following [KM3]) some facts about the polynomial q-expansions which arise as component functions of holomorphic vector-valued modular forms which will be needed for the proof of the main Theorems.

Let (ρ, F) be a holomorphic vector-valued modular form of weight k. Replacing (ρ, F) by an equivalent vector-valued modular form if necessary, we may, and shall, assume that $\rho(T)$ is in (modified) Jordan canonical form.[2] In passing from (ρ, F) to an equivalent vector-valued modular form, the component functions of F are replaced by linear combinations of the components, but this will not matter to us. Let the ith Jordan block of $\rho(T)$ have size m_i, and label the corresponding component functions of $F(\tau)$ as $\varphi_1^{(i)}(\tau), \ldots, \varphi_{m_i}^{(i)}(\tau)$. By [KM3] they have polynomial q-expansions

$$\varphi_l^{(i)}(\tau) = \sum_{s=0}^{l-1} \binom{\tau}{s} h_{l-1-s}^{(i)}(\tau) \quad (1 \leq l \leq m_i), \tag{2}$$

each $h_s^{(i)}(\tau)$ having a left-finite q-series

$$h_s^{(i)}(\tau) = e^{2\pi i \mu_i \tau} \sum_{n=\nu_i}^{\infty} a_n(s, i) e^{2\pi i n \tau} \quad (0 \leq s \leq m_i - 1, \ \nu_i \in \mathbb{Z}). \tag{3}$$

Here, $e^{2\pi i \mu_i}$ is the eigenvalue of $\rho(T)$ determined by the ith block and $0 \leq \mu_i < 1$. (It is here that we are using the assumption that the eigenvalues of $\rho(T)$ have absolute

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[2]A minor variant of the usual Jordan canonical form. See [KM3] for details.

value 1.) $F(\tau)$ is called *holomorphic* if, for each Jordan block, each q-series $h_s^{(i)}(\tau)$ has only *nonnegative* powers of q, i.e., $a_n(s, i) = 0$ whenever $n + \mu_i < 0$.

3   Proof of the main Theorems

In this Section we will prove Theorem 2. As we have explained, this is equivalent to Theorem 1.

In order to prove Theorem 2, we may replace (ρ, F) by any equivalent vector-valued modular form. Thus we may, and from now on shall, assume without loss that $\rho(T)$ is in (modified) Jordan canonical form. We assume that $\rho(T)$ has t Jordan blocks, which we may, and shall, further assume are ordered in decreasing size $M = m_1 \geq m_2 \geq \ldots \geq m_t$. Thus $m_1 + \ldots + m_t = p$, and we may speak, with an obvious meaning, of the component functions in a block. The ith. block corresponds to an eigenvalue $e^{2\pi i \mu_i}$ of $\rho(T)$, and we let the component functions of $F(\tau)$ in that block be as in (2), (3).

Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{4}$$

Because (ρ, F) is a vector-valued modular form of weight k we have

$$\rho(\gamma)F(\tau) = (c\tau + d)^{-k}F(\gamma\tau).$$

So if $\varphi_u(\tau) = \varphi_v^{(i)}(\tau)$ is the uth component of the ith block of $F(\tau)$ (so that $u = m_1 + \ldots + m_{i-1} + v$), then

$$(c\tau + d)^{-k}\varphi_u(\gamma\tau) = \sum_{j=1}^{t}\sum_{l=1}^{m_j} \alpha_l^{(j)}\varphi_l^{(j)}(\tau)$$

where $(\ldots, \underbrace{\alpha_1^{(j)}, \ldots, \alpha_{m_j}^{(j)}}_{j\text{th block}}, \ldots)$ is the uth row of $\rho(\gamma)$. Using (2), we obtain

$$
\begin{aligned}
(c\tau + d)^{-k}\varphi_u(\gamma\tau) &= \sum_{j=1}^{t}\sum_{l=1}^{m_j}\sum_{s=0}^{l-1} \alpha_l^{(j)}\binom{\tau}{s}h_{l-1-s}^{(j)}(\tau) \\
&= \sum_{s=0}^{M-1}\binom{\tau}{s}\left(\sum_{j=1}^{t}\sum_{l=s+1}^{m_j} \alpha_l^{(j)}\, h_{l-1-s}^{(j)}(\tau)\right) \\
&= \sum_{s=0}^{M-1}\binom{\tau}{s}\sum_{j=1}^{t}\left(\alpha_{s+1}^{(j)}h_0^{(j)}(\tau) + \sum_{l=s+2}^{m_j} \alpha_l^{(j)}\, h_{l-1-s}^{(j)}(\tau)\right).
\end{aligned}
$$

(Here, $\alpha_{s+1}^{(j)} = 0$ if $s \geq m_j$.)

Because the component functions of (ρ, F) are linearly independent, $\varphi_u(\tau)$ is nonzero and the previous display is not identically zero. So there is a largest integer B in the range $0 \le B \le M - 1$ such that the summand corresponding to $\binom{\tau}{B}$ does *not* vanish. Now note that $\varphi_1^{(j)}(\tau) = h_0^{(j)}(\tau)$. Because the component functions are linearly independent, then in particular the $h_0^{(j)}(\tau)$ are linearly independent, and we can conclude that

$$\alpha_{s+1}^{(j)} = 0 \quad (1 \le j \le t, \ s > B + 1), \tag{5}$$
$$\alpha_{B+1}^{(j)} \ \text{ are not all zero } \ (1 \le j \le t).$$

It follows that

$$(c\tau + d)^{-k}\varphi_u(\gamma\tau)$$
$$= \binom{\tau}{B} \sum_{j=1}^{t} \alpha_{B+1}^{(j)} h_0^{(j)}(\tau) + \sum_{s=0}^{B-1} \binom{\tau}{s} \sum_{j=1}^{t} \sum_{l=s+2}^{m_j} \alpha_l^{(j)} \, h_{l-1-s}^{(j)}(\tau) \tag{6}$$

and the first term on the right hand side of (6) is nonzero. Incorporating (3), we obtain

$$(c\tau + d)^{-k}\varphi_u(\gamma\tau)$$
$$= \binom{\tau}{B} \sum_{j=1}^{t} \alpha_{B+1}^{(j)} e^{2\pi i \mu_j \tau} \sum_{n=\nu_j}^{\infty} a_n(0, j) e^{2\pi i n \tau}$$
$$+ \sum_{s=0}^{B-1} \binom{\tau}{s} \sum_{j=1}^{t} \sum_{l=s+2}^{m_j} \sum_{n=\nu_j}^{\infty} \alpha_l^{(j)} e^{2\pi i \mu_j \tau} a_n(l - 1 - s, j) e^{2\pi i n \tau}. \tag{7}$$

Let $\tilde{\mu}_1, \ldots, \tilde{\mu}_p$ be the *distinct* values among $\mu_1, \ldots, \mu_t$. Then we can rewrite (7) in the form

$$(c\tau + d)^{-k}\varphi_u(\gamma\tau)$$
$$= \binom{\tau}{B} \sum_{j=1}^{p} e^{2\pi i \tilde{\mu}_j \tau} g_B^{(j)}(\tau) + \sum_{s=0}^{B-1} \binom{\tau}{s} \sum_{j=1}^{p} e^{2\pi i \tilde{\mu}_j \tau} g_s^{(j)}(\tau), \tag{8}$$

where the first term in (8) is *nonzero* and each $g_m^{(j)}(\tau)$ is a left-finite *pure q-series*, i.e. one with only integral powers of q.

Consider the *nonzero* summands

$$e^{2\pi i \tilde{\mu}_j \tau} g_B^{(j)}(\tau) = \sum_{n=\nu(j,B)}^{\infty} b_n(j, B) q^{n + \tilde{\mu}_j} \tag{9}$$
$$= b_{\nu(j,B)}(j, B) q^{\nu(j,B) + \tilde{\mu}_j}(1 + \text{positive integral powers of } q)$$

that occur in the first term on the right hand side of (8). Let J be the corresponding set of indices j. Because the $\tilde{\mu}_j$ are distinct, there is a *unique* $j_0 \in J$ which minimizes the expression

$$\nu(j, B) + \tilde{\mu}_j.$$

Let $J' = J \setminus \{j_0\}$. Hence, there is $y_0 > 0$ such that for $\Im(\tau) > y_0$ we have

$$\left| e^{2\pi i \tilde{\mu}_{j_0} \tau} g_B^{(j_0)}(\tau) \right| > 2 \left| \sum_{j \in J'} e^{2\pi i \tilde{\mu}_j \tau} g_B^{(j)}(\tau) \right|.$$

Taking into account the terms $e^{2\pi i \tilde{\mu}_j \tau} g_B^{(j)}(\tau)$ that vanish, we obtain for $\Im(\tau) > y_0$:

$$\left| \sum_{j=1}^{p} e^{2\pi i \tilde{\mu}_j \tau} g_B^{(j)}(\tau) \right| > \left| e^{2\pi i \tilde{\mu}_{j_0} \tau} g_B^{(j_0)}(\tau) \right| - \left| \sum_{j \in J'} e^{2\pi i \tilde{\mu}_j \tau} g_B^{(j)}(\tau) \right|$$

$$> 1/2 \left| e^{2\pi i \tilde{\mu}_{j_0} \tau} g_B^{(j_0)}(\tau) \right| > 0. \tag{10}$$

In (10), for $N \in \mathbb{Z}$ we have $\Im(\tau + N) = \Im(\tau) > y_0$. So (10) holds with τ replaced by $\tau + N$. Because $g_B^{(j)}(\tau + N) = g_B^{(j)}(\tau)$, we see that

$$\left| \sum_{j=1}^{p} e^{2\pi i \tilde{\mu}_j (\tau + N)} g_B^{(j)}(\tau) \right| > 1/2 \left| e^{2\pi i \tilde{\mu}_{j_0} \tau} g_B^{(j_0)}(\tau) \right| > 0 \quad (N \in \mathbb{Z}, \ \Im(\tau) > y_0). \tag{11}$$

At this point we return to (8). Replace τ by $\tau + N$ to obtain

$$(c\tau + cN + d)^{-k} \varphi_u(\gamma(\tau + N))$$

$$= \binom{\tau + N}{B} \sum_{j=1}^{p} e^{2\pi i \tilde{\mu}_j (\tau + N)} g_B^{(j)}(\tau) + \sum_{s=0}^{B-1} \binom{\tau + N}{s} \sum_{j=1}^{p} e^{2\pi i \tilde{\mu}_j (\tau + N)} g_s^{(j)}(\tau). \tag{12}$$

Set

$$\Sigma_1(\tau) = \Sigma_1(\tau; \gamma) = \sum_{s=0}^{B} \binom{\tau}{s} \sum_{j=1}^{p} e^{2\pi i \tilde{\mu}_j (\tau)} g_s^{(j)}(\tau),$$

$$\Sigma_2(\tau, N) = \Sigma_2(\tau, N; \gamma) = \Sigma_1(\tau + N).$$

Thus (12) reads

$$\varphi_u(\gamma(\tau + N)) = (c\tau + cN + d)^k \Sigma_2(\tau, N). \tag{13}$$

Next we examine the powers of N that appear in $\Sigma_2(\tau, N)$. Now

$$\binom{\tau + N}{s} = \frac{1}{s!}(\tau + N)(\tau + N - 1)\ldots(\tau + N - s + 1)$$

$$= \frac{N^s}{s!} + O(N^{s-1}), \quad N \to \infty.$$

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Therefore, the highest power of N occurring with nonzero coefficient in $\Sigma_2(\tau, N)$ is N^B, the coefficient in question being

$$\frac{1}{B!} \sum_{j=1}^{p} e^{2\pi i \tilde{\mu}_j (\tau+N)} g_B^{(j)}(\tau). \tag{14}$$

Hence, we obtain

$$\Sigma_2(\tau, N) = \frac{N^B}{B!} \sum_{j=1}^{p} e^{2\pi i \tilde{\mu}_j (\tau+N)} g_B^{(j)}(\tau) + O(N^{B-1}), \quad N \to \infty, \tag{15}$$

with nonzero leading coefficient (14).

So far, the component function $\varphi_u(\tau)$ of $F(\tau)$ has been arbitrary. Now we claim that there is at least one component such that the integer B occurring in (15), and thereby also in (13), is equal to $M - 1$. Indeed, because the first block has size M, the Mth. component $\varphi_M^{(1)}(\tau)$ of $F(\tau)$ has the polynomial $\binom{\tau}{M-1}$ occurring in its logarithmic q-expansion (2) with *nontrivial* coefficient $h_0^{(1)}(\tau)$. Because $\rho(\gamma)$ is nonsingular, at least one row of $\rho(\gamma)$, say the uth., has a nonzero value $\alpha_M^{(1)}$ in the Mth. column. Thanks to (5) we must have $B + 1 = M$, as asserted.

With $\varphi_u(\tau)$ as in the last paragraph, we have for $N \to \infty$,

$$\varphi_u(\gamma(\tau+N)) =$$
$$(c\tau + cN + d)^k \left(\frac{N^{M-1}}{(M-1)!} \sum_{j=1}^{p} e^{2\pi i \tilde{\mu}_j (\tau+N)} g_{M-1}^{(j)}(\tau) + O(N^{M-2}) \right). \tag{16}$$

Lemma 3.1 *Let $\varphi_u(\tau)$ be as before, and suppose that there exists a rational number a/c, $((a,c) = 1, c \neq 0)$ at which $\varphi_u(\tau)$ is continuous from above. Then $k \leq 1 - M$.*

Proof: First note that

$$\gamma(\tau + N) = \frac{a + b/(\tau+N)}{c + d/(\tau+N)} \to a/c \text{ as } N \to \infty \text{ within } \mathfrak{H}.$$

By the continuity assumption of the Lemma, $\varphi_u(\tau)$ remains bounded as $N \to \infty$. Choosing y_0 large enough, we see from (11) that

$$\left| \sum_{j=1}^{p} e^{2\pi i \tilde{\mu}_j (\tau+N)} g_{M-1}^{(j)}(\tau) \right|$$

is bounded away from 0 as $N \to \infty$. Because $c \neq 0$, we deduce that the right hand side of (16) is $\approx \alpha(N) N^{k+M-1}$ with $\alpha(N) \neq 0$. If $k > 1 - M$ this implies that the right hand side is unbounded as $N \to \infty$. This contradiction proves the Lemma. $\square$

Lemma 3.2 *Let $\varphi_u(\tau)$ be as in (16), and suppose that $\varphi_u(\tau)$ is holomorphic in a region containing $\mathfrak{H} \cup \mathfrak{I}$ with $\mathfrak{I}$ a non-empty open interval in $\mathbb{R}$. Then $k \geq 1 - M$.*

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Proof: Choose rational a/c as in the last Lemma so that $a/c \in \mathfrak{I}$. The argument of the previous Lemma shows that the right hand side of (16) is $\approx \alpha(N)N^{k+M-1}$ with $\alpha(N) \neq 0$. Indeed, we easily see from (16) that $\alpha(N)$ has an upper bound independent of N for $N \to \infty$. Then if $k < 1 - M$ the right hand side of (16) $\to 0$ as $N \to \infty$.

On the other hand, the left hand side of (16) $\to \varphi_u(a/c)$ as $N \to \infty$. We conclude that $\varphi_u(a/c) = 0$, and because this holds for all rationals in $\mathfrak{I}$ then $\varphi_u(\tau)$ is identically zero thanks to the regularity assumption on $\varphi_u(\tau)$. Because the components of $F(\tau)$ are linearly independent $\varphi_u(\tau)$ cannot vanish, and this contradiction shows that $k \geq 1 - M$, as required. $\qquad\square$

Proposition 3.3 *Assume that the regularity assumption of Lemma 3.2 applies to* all *components of $F(\tau)$. Then $k = 1 - M$ and each component is a polynomial of degree at most $M - 1$.*

Proof: Because of the regularity assumptions of the present Proposition we may apply Lemmas 3.1 and 3.2 to find that $k = 1 - M$. If, for some component $\varphi_u(\tau)$, the integer B that occurs in (15) is *less* than $M - 1$, the argument of Lemma 3.1 yields a contradiction. Since, in any case, we have $B \leq M - 1$, then in fact $B = M - 1$ for all components. As a result, (16) holds for *every* component $\varphi(\tau)$ of $F(\tau)$.

Differentiate (8) (now with $B = M-1$) M times and apply the well-known identity of Bol [B]:

$$D^{(M)}\left((c\tau + d)^{M-1}\varphi(\gamma\tau)\right) = (c\tau + d)^{-1-M}\varphi^{(M)}(\gamma\tau).$$

We obtain (using Leibniz's rule) for $|\tau| \to \infty$ that

$$\varphi^{(M)}(\gamma\tau)$$
$$= (c\tau + d)^{M+1}D^{(M)}\left(\sum_{s=0}^{M-1}\binom{\tau}{s}\sum_{j=1}^{p}e^{2\pi i\tilde{\mu}_j\tau}g_s^{(j)}(\tau)\right)$$
$$= (c\tau + d)^{M+1}\left(\frac{\tau^{M-1}}{(M-1)!}D^{(M)}\left(\sum_{j=1}^{p}e^{2\pi i\tilde{\mu}_j\tau}g_{M-1}^{(j)}(\tau)\right) + O(|\tau|^{M-2})\right).$$

Therefore, for $N \to \infty$,

$$\varphi^{(M)}(\gamma(\tau + N)) = \qquad\qquad (17)$$
$$(c\tau + cN + d)^{M+1}\left(\frac{(\tau + N)^{M-1}}{(M-1)!}D^{(M)}\left(\sum_{j=1}^{p}e^{2\pi i\tilde{\mu}_j(\tau+N)}g_{M-1}^{(j)}(\tau)\right) + O(N^{M-2})\right).$$

Take $c \neq 0$ with γ as in (4) and $a/c \in \mathfrak{I}$, and apply the regularity assumption to $\varphi(\tau)$. Then the left hand side of (17) has a limit $\varphi^{(M)}(a/c)$ for $N \to \infty$. On the other hand, we know that $\left|\sum_{j=1}^{p}e^{2\pi i\tilde{\mu}_j(\tau+N)}g_{M-1}^{(j)}(\tau)\right|$ is bounded away from zero. So

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if $D^{(M)}\left(\sum_{j=1}^{p} e^{2\pi i \tilde{\mu}_j (\tau + N)} g_{M-1}^{(j)}(\tau)\right)$ does not vanish identically then the right hand side of (17) is $\approx \alpha(N)N^{2M}$ for $N \to \infty$, with $\alpha(N)$ bounded away from zero. This contradiction shows that in fact

$$D^{(M)}\left(\sum_{j=1}^{p} e^{2\pi i \tilde{\mu}_j (\tau + N)} g_{M-1}^{(j)}(\tau)\right) \equiv 0$$

From (9), we have

$$D^{(M)}\left(\sum_{j=1}^{p} e^{2\pi i \tilde{\mu}_j (\tau + N)} g_{M-1}^{(j)}(\tau)\right)$$

$$= \sum_{j=1}^{p} e^{2\pi i \tilde{\mu}_j N} \sum_{n=\nu(j,M-1)}^{\infty} b_n(j, M-1)(2\pi i(n + \tilde{\mu}_j))^M q^{n+\tilde{\mu}_j},$$

so that

$$\sum_{j=1}^{p} e^{2\pi i \tilde{\mu}_j N} b_n(j, M-1)(n + \tilde{\mu}_j)^M = 0 \quad (n \geq \nu(j, M-1)).$$

This implies that $b_n(j, M-1) = 0$ whenever $n + \tilde{\mu}_j \neq 0$. Because $b_{\nu(j,M-1)}(j, M-1) \neq 0$, we must have

$$\tilde{\mu}_j = 0 \quad (1 \leq j \leq p), \tag{18}$$

which amounts to the assertion that all $\mu_j = 0$ $(1 \leq j \leq t)$. Moreover $b_n(j, M-1) = 0$ for $n \neq 0$, so that

$$g_{M-1}^{(j)}(\tau) = b_0(j, M-1)$$

is constant. Now (8) reads

$$(c\tau + d)^{M-1}\varphi(\gamma\tau) = \binom{\tau}{M-1}\sum_{j=1}^{p} b_0(j, M-1) + \sum_{s=0}^{M-2}\binom{\tau}{s}\sum_{j=1}^{p} g_s^{(j)}(\tau). \tag{19}$$

We now repeat the argument $M-1$ times, starting with (19) in place of (8). We end up with an identity of the form

$$(c\tau + d)^{M-1}\varphi(\gamma\tau) = \sum_{s=0}^{M-1}\binom{\tau}{s}\sum_{j=1}^{p} b_0(j, s),$$

where of course the right hand side is a polynomial $p(\tau)$ of degree at most $M-1$. Then

$$\begin{aligned}
\varphi(\tau) &= (c\gamma^{-1}\tau + d)^{1-M} p\left(\frac{d\tau - b}{-c\tau + a}\right) \\
&= (c\tau + d)^{M-1} p\left(\frac{d\tau - b}{-c\tau + a}\right)
\end{aligned}$$

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is itself a polynomial of degree at most $M-1$. This completes the proof of Proposition 3.3. □

It is now easy to complete the proof of Theorem 2. It is only necessary to establish the implication (a) $\Rightarrow$ (b). Assuming (a) means that Proposition 3.3 is applicable, so that we have $k = 1 - M$ and the components of $F(\tau)$ are polynomials of degree at most $M - 1$. Because the components are linearly independent, it must be the case that the maximal block size M is equal to the dimension p of the representation ρ. Thus $k = 1 - p$, and the component functions *span* the space of polynomials of degree at most $p - 1$. The fact that (ρ, F) is equivalent to (σ, C) then follows from the discussion in Section 2.

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